PHYS 705: Classical Mechanics

Energy Conservation and Time Invariance

Note the distinction:

$f(q,\dot{q},t)$ is time invariant $\neq$ $f(q,\dot{q},t)$ is a constant in time

$$\frac{\partial f}{\partial t}=0$$

⇓

functional form of f does NOT change with a time shift:

$$t \rightarrow t+\Delta$$

f can still depends on time *implicitly* thru $(q,\dot{q})$

$$\frac{df}{dt}=0$$

⇓

f has a constant value in time

Energy Conservation and Time Invariance in Configuration Space

Conservation of *h* (Jacobi Integral):

shown in class (check class note)

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad h \text{ is conserved!}$$

But, in general $\dfrac{\partial h, E}{\partial t} = 0 \quad \not\Rightarrow \quad$ *h, E* is conserved

Or, vice versa, $\dfrac{\partial h, E}{\partial t} \neq 0 \quad \not\Rightarrow \quad$ *h, E* is NOT conserved

recall in general… $\left[\dfrac{\partial h, E}{\partial t} \neq \dfrac{dh, E}{dt}\right]$

Energy Conservation and Time Invariance in Phase Space

Conservation of H (Hamiltonian):

Using the Hamilton's Equation, we explicitly showed that:

$$\frac{dH\left(q,p\right)}{dt}=\frac{\partial H\left(q,p\right)}{\partial t}$$

It is an important property of dynamics in Phase Space.

So, if $\frac{\partial H}{\partial t}=0$ ⇒ $\frac{dH}{dt}=0$ and H is conserved!

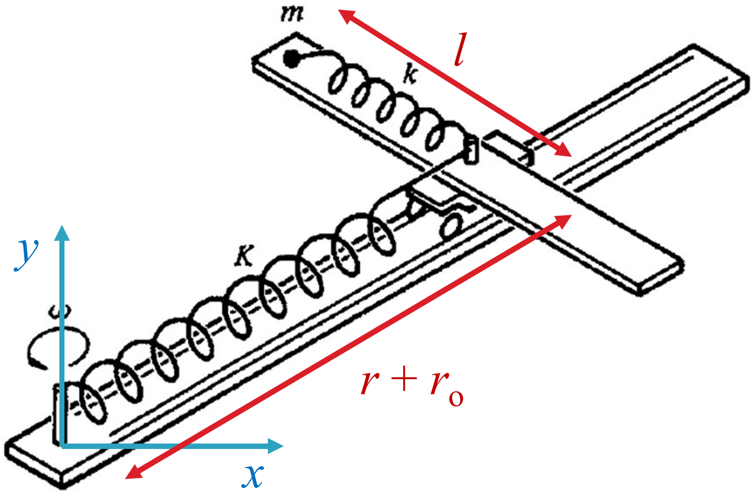
This is true for $H\left(q,p\right)$ in Phase Space and not for $h\left(q,\dot{q}\right)$ Configuration Space!

Energy Conservation and Time Invariance

HW#5 2.21

(x, y) Lab Frame (fixed)

(r, l) Rotating Frame

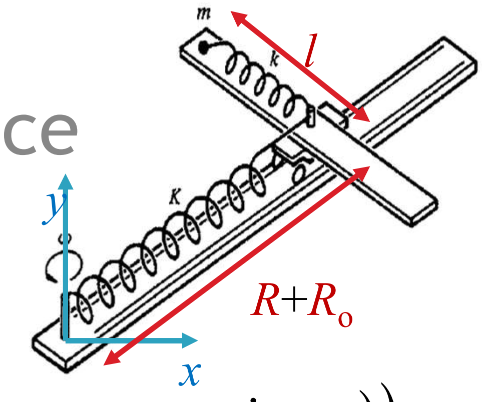


Point Transform between Lab and Rotating frames

$$\begin{cases} x = (r + r_o)\cos\omega t - l\sin\omega t \\ y = (r + r_o)\sin\omega t + l\cos\omega t \end{cases}$$

$$\begin{cases} r = x\cos\omega t + y\sin\omega t - r_o \\ l = -x\sin\omega t + y\cos\omega t \end{cases}$$

Energy Conservation and Time Invariance



Lagrangian:

$$L(x,y,t)=T-V=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)-\frac{k}{2}\left(x^2+y^2+r_o^2-2r_o\left(x\cos\omega t+y\sin\omega t\right)\right)$$

$$L(r,l,t)=T-V=\frac{m}{2}\left(\left(\left(r_o+r\right)\omega+\dot{l}\right)^2+\left(\dot{r}-l\omega\right)^2\right)-\frac{k}{2}\left(r^2+l^2\right)$$

Total Energy:

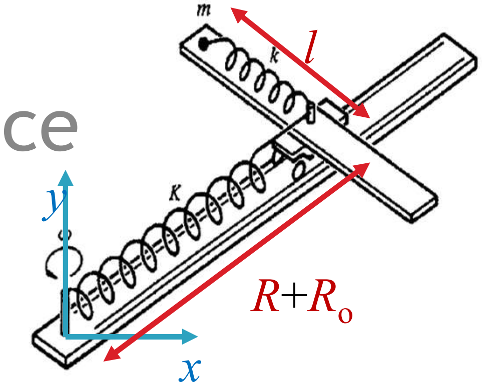
$$E(x,y,t)=T+V=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{k}{2}\left(x^2+y^2+r_o^2-2r_o\left(x\cos\omega t+y\sin\omega t\right)\right)$$

$$E(r,l,t)=T+V=\frac{m}{2}\left(\left(\left(r_o+r\right)\omega+\dot{l}\right)^2+\left(\dot{r}-l\omega\right)^2\right)+\frac{k}{2}\left(r^2+l^2\right)$$

Under a coord. transf., total energy of system CANNOT change

$$E(x,y,t)=E\left(r,l,t\right)=T+V$$

Energy Conservation and Time Invariance



Jacobi Integral in the Lab Frame:

$$h(x,y,t)=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{k}{2}\left(x^2+y^2+r_o^2-2r_o\left(x\cos\omega t+y\sin\omega t\right)\right)$$

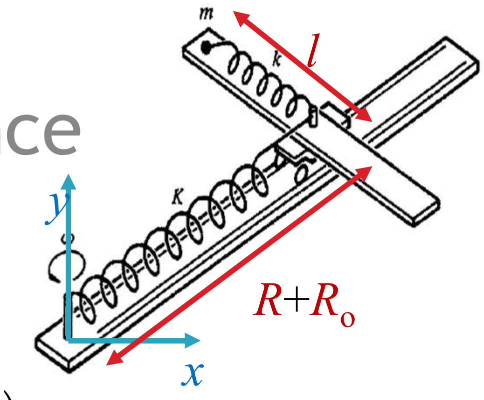
Since $V(x,y)$ does not depend on $(\dot{x},\dot{y})$ and $(x,y)\mapsto(x,y)$, identity trans does not depends on time, so

$$h(x,y,t)=E(x,y,t)$$

Now, since $\frac{\partial L(x,y,t)}{\partial t}\neq 0, \frac{dh(x,y,t)}{dt}\neq 0$

So, $h(x,y,t)=E(x,y,t)$ is NOT conserved in the Lab frame.

Energy Conservation and Time Invariance



Jacobi Integral in Rotating Frame:

$$h(r,l)=\frac{m}{2}\left(\dot{r}^2+\dot{l}^2\right)+\frac{k}{2}\left(r^2+l^2\right)-\frac{m}{2}\left(\left(r_o+r\right)^2+l^2\right)\omega^2$$

Now, since $(x,y)\mapsto(r,l)$ depends on time explicitly,

$h(r,l)\neq E(r,l,t)$,i.e., Jacobi Integral in (r,l) is not the total energy.

However, now since $\dfrac{\partial L(r,l,t)}{\partial t}=0$, $\dfrac{dh(r,l)}{dt}=0$

So, $h(r,l)$ is conserved in the Rotating frame but it is NOT the total energy.

Energy Conservation and Time Invariance

$$E\left(r,l,t\right)=T+V=\frac{m}{2}\left(\left(\left(r_o+r\right)\omega+\dot{l}\right)^2+\left(\dot{r}-l\omega\right)^2\right)+\frac{k}{2}\left(r^2+l^2\right)$$

Here is an example where

$$\frac{\partial E\left(r,l,t\right)}{\partial t}=0$$

BUT, $E\left(r,l,t\right)=E\left(x,y,t\right)$ still varies in time (not conserved)

$$\frac{dE}{dt}\neq 0$$

Energy Conservation and Time Invariance

For the converse, consider just a mass m at the end of a spring with spring constant k moving in 1D

The Lagrangian is given by $L(x) = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2$

The total energy $E\left(x\right) = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2$ is conserved.

Now, let say, the system is observed in a moving frame with respect to the fixed frame with a constant speed a . The generalized coordinate in the moving frame is

$$q = x + at$$

Energy Conservation and Time Invariance

The Lagrangian in the moving frame is now explicitly depended on time,

$$L(q,t)=\frac{m}{2}(\dot{q}-a)^2-\frac{k}{2}(q-at)^2$$

The Total Energy in q is also explicitly dependent on time,

$$E(q,t)=\frac{m}{2}(\dot{q}-a)^2+\frac{k}{2}(q-at)^2$$

By a direct back substitution with $x=q-at$ and $\dot{x}=\dot{q}-a$, the total energy is still the total energy,

$$E(q,t)\xrightarrow[q\to x]{}\frac{m}{2}\dot{x}^2+\frac{k}{2}x^2=E(x,t)$$

So, E is still conserved. So, here is the example, while we have

$$\frac{\partial E\left(q,t\right)}{\partial t}\neq 0 \qquad \text{but} \qquad \frac{dE}{dt}=0$$

L, and h/E conservation (HW #5 2.21)

1. To analyze conservation of *h*, recall that we assume the homogeneous form of the EL equation with a set of *proper* generalized coordinates, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0$$

So, we need to start with an *L* with a set of proper generalized coordinates.

2. *h* = *E* are not necessary the same!

Only, if $U=U(q_i)$ AND $\frac{\partial \mathbf{r}_i}{\partial t}=0$, then *h* = *E*.

3. Total energy *E* does not change under a coordinate transformation

But, the conservation of Jacobi Integral *h* will depend on a given coordinate system!

Review

Legendre Transform

Here is the definition of the Legendre Transform for $F\left(x\right)$

$$G\left(s\right)=sx\left(s\right)-F\left(x\left(s\right)\right)$$

Note that $G\left(s\right)$ is a function of s and we have to express $F\left(x\left(s\right)\right)$ in terms of s by inverting the relation: $s=\dfrac{dF\left(x\right)}{dx}$ to get $x\left(s\right)$

$G\left(s\right)$ is used when $s=\dfrac{dF\left(x\right)}{dx}$ is more preferred over x itself as the variable of choice in the analysis.

Hamiltonian Formulation

- Instead of using the Lagrangian, $L = L\left(q_j, \dot{q}_j, t\right)$, we will introduce a new function that depends on q_j, p_j, and t: $H = H\left(q_j, p_j, t\right)$

- This new function is call the Hamiltonian and it is defined by:

$$H = \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L$$

(Einstein' s Convention:

Repeated indices are summed)

- Plugging in the definition for the generalized momenta: $p_j \equiv \dfrac{\partial L}{\partial \dot{q}_j}$

$$H = p_j \dot{q}_j - L \qquad (sum)$$

- $H\left(q_j, p_j, t\right)$ is the Legendre Transform of $L\left(q_j, \dot{q}_j, t\right)$ with $\left(q_j, \dot{q}_j\right) \to \left(q_j, p_j\right)$

$x \to \dot{q} \qquad s \to \partial L / \partial \dot{q} \equiv p$

$F \to L \qquad G \to H$

$$G = sx - F \quad \to \quad H = p\dot{q} - L$$

Hamiltonian Formulation

- Starting with $H = p_j \dot{q}_j - L$

- Taking the differential,

$$dH = p_j d\dot{q}_j + \dot{q}_j dp_j - dL \qquad (sum)$$

- Then, we require that $H = H\left(q_j, p_j, t\right)$ so that we should have

$$dH = \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt \qquad (sum)$$

- Equating the two expressions for dH and applying the EL equation

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \dot{p}_j$$

to expand out the dL term…

Hamiltonian Formulation

Crank crank crank and dot dot dot, we get:

$$\begin{cases} \dfrac{\partial H}{\partial q_j} = -\dot{p}_j \\ \dfrac{\partial H}{\partial p_j} = \dot{q}_j \end{cases} \quad \text{and} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

These are called the Hamilton's Equations of Motion and they are the desired set of equations giving the EOM in phase space.

Energy Conservation and Time Invariance in Phase Space

By taking the full-time derivative of $H(q,p,t)$ and using the Hamilton's Equation, we explicitly showed that:

$$\frac{dH(q,p,t)}{dt} = \frac{\partial H(q,p,t)}{\partial t}$$

It is an important property of dynamics in Phase Space.

So, if $\frac{\partial H}{\partial t} = 0 \quad \Rightarrow \quad \frac{dH}{dt} = 0$ and H is conserved!

This is true for $H(q,p,t)$ in Phase Space and not for $h(q,\dot{q},t)$ Configuration Space!

Hamiltonian Equations in Matrix (Symplectic) Notation

The Hamilton equation can then be written in a compact form:

$$\boxed{\dot{\boldsymbol{\eta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\eta}}}$$

if we define a $2n \times 2n$ anti-symmetric matrix $\mathbf{J}$,

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$$

where **I** is a $n \times n$ identity matrix

0 is a $n \times n$ zero matrix

Note that the transpose of **J** is its own inverse (orthogonal):

$$\mathbf{J}^T = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \qquad \text{and} \qquad \mathbf{J}^T\mathbf{J} = \mathbf{J}\mathbf{J}^T = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

This is typically referred to as the matrix (or symplectic) notation for the Hamilton equations.

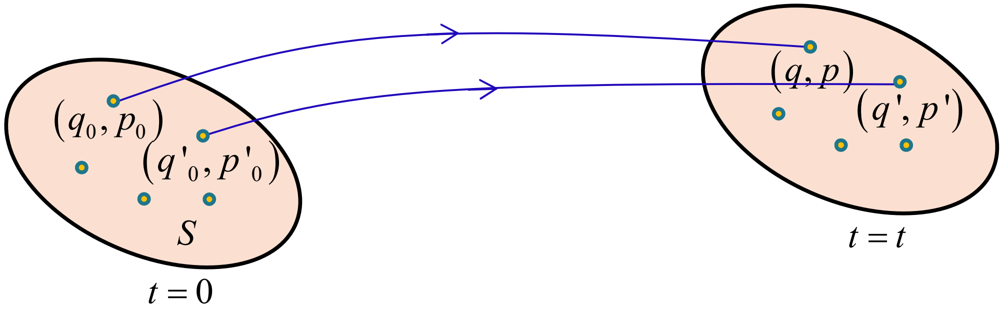
Connection to Statistical Mechanics

- The Hamilton's Equations describe motion in phase space

- a point in phase space $\left(q_j, p_j\right)$ uniquely determines the state of the system AND its future evolution.

- points nearby represent system states with similar but slightly different initial conditions.

- One can imagine a *cloud of points (ensemble of systems)* bounded by a closed surface *S* with nearly identical initial conditions moving in time.



Liouville's Theorem

One can show as a direct consequence of the Hamilton Equation of Motion that the phase space volume of this cloud of points (ensemble of system) is conserved!

$$\frac{dV}{dt} = 0\,!$$

This is the Liouville's Theorem: collection of phase-space points move as an incompressible fluid.

→ Phase space volume occupied by a set of points in phase space is *constant* in time.

Canonical Transformation

Recall (from hw) that the Euler-Lagrange Equation is invariant for a **point transformation**: $Q_j = Q_j(q,t)$

i.e., if we have,
$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0,$$

then,
$$\frac{\partial L}{\partial Q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_j}\right) = 0,$$

Now, the idea is to find a generalized (**canonical**) transformation in *phase space* (not config. space) such that the Hamilton's Equations are invariant !

$$Q_j = Q_j(q, p, t)$$
$$P_j = P_j(q, p, t)$$

(In general, we look for transformations which are *invertible*.)

Canonical Transformation

We need to find the appropriate (canonical) transformation

$$Q_j = Q_j(q, p, t) \quad \text{and} \quad P_j = P_j(q, p, t)$$

such that there exist a transformed Hamiltonian $K(Q, P, t)$

with which the Hamilton's Equations are satisfied:

$$\dot{Q}_j = \frac{\partial K}{\partial P_j} \quad and \quad \dot{P}_j = -\frac{\partial K}{\partial Q_j}$$

(The form of the EOM must be *invariant* in the new coordinates.)

** It is important to further stated that the transformation considered must also be *problem-independent* meaning that (Q, P) must be canonical coordinates for all system with the same number of dofs.

Canonical Transformation

To see what this condition might say about our canonical transformation, we need to go back to the Hamilton's Principle:

Hamilton's Principle: The motion of the system in *configuration space* is such that the action *I* has a stationary value for the actual path, .i.e.,

$$\delta I = \delta \int_{t_1}^{t_2} L dt = 0$$

Now, we need to extend this to the 2*n*-dimensional *phase space*

1. The integrant in the action integral must now be a function of the independent conjugate variable q_j, p_j and their derivatives $\dot{q}_j, \dot{p}_j$
2. We will consider variations in all 2*n* phase space coordinates

Hamilton’s Principle in Phase Space

To rewrite the integrant in terms of $q_j, p_j, \dot{q}_j, \dot{p}_j$, we will utilize the definition for the Hamiltonian (or the inverse Legendre Transform):

$$H = p_j \dot{q}_j - L \quad \rightarrow \quad L = p_j \dot{q}_j - H(q, p, t) \qquad \text{(Einstein's sum rule)}$$

Substituting this into our variation equation, we have

$$\delta I = \delta \int_{t_1}^{t_2} L dt = \delta \int_{t_1}^{t_2} \left[p_j \dot{q}_j - H(q, p, t) \right] dt = 0$$

Hamilton’s Principle in Phase Space

Applying the Hamilton’s Principle in Phase Space, the resulting dynamical equation is the Hamilton’s Equations.

$$\begin{cases} \dot{p}_j = -\dfrac{\partial H}{\partial q_j} \\ \dot{q}_j = \dfrac{\partial H}{\partial p_j} \end{cases}$$

Canonical Variables and Hamiltonian Formalism

In the Hamiltonian Formulation of Mechanics,

→ q_j, p_j are independent variables in phase space on equal footing

→ The Hamilton's Equation for q_j, p_j are "symmetric" (*symplectic*)

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad and \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

→ This elegant formal structure of mechanics affords us the freedom in selecting other appropriate canonical variables as our phase space "coordinates" and "momenta"

- As long as the new variables formally satisfy this abstract structure (the form of the Hamilton's Equations).

Canonical Transformation

When is a transformation to Q, P canonical?

→ We need Hamilton's Equations to hold in both systems

This means that we need to have the following variational conditions:

$$\delta\int\left[p_j\dot{q}_j - H(q,p,t)\right]dt = 0 \quad \text{AND} \quad \delta\int\left[P_j\dot{Q}_j - K(Q,P,t)\right]dt = 0$$

→ For this to be true simultaneously, the integrands must equal

→ And, as argued in our previous lecture, this is also true if they are differed by a full-time derivative of a function of *any* of the phase space variables involved + time:

$$\Rightarrow \quad \boxed{p_j\dot{q}_j - H(q,p,t) = P_j\dot{Q}_j - K(Q,P,t) + \frac{dF}{dt}\left(q,p,Q,P,t\right)}$$

Canonical Transformation

$$p_j\dot{q}_j - H(q,p,t) = P_j\dot{Q}_j - K(Q,P,t) + \frac{dF}{dt}\left(q,p,Q,P,t\right) \quad (1)\ (G9.11)$$

F is called the *Generating Function* for the canonical transformation:

$$(q_j,p_j) \to \left(Q_j,P_j\right):\ \begin{cases} Q_j = Q_j(q,p,t) \\ P_j = P_j(q,p,t) \end{cases}$$

→ Depending on the form of the generating functions (which pair of canonical variables being considered as the *independent* variables for the Generating Function), we can classify canonical transformations into four basic types.

$$Q_j = Q_j(q,p,t)$$
$$P_j = P_j(q,p,t)$$

Canonical Transformation: 4 Types

$$p_j \dot{q}_j - H(q,p,t) = P_j \dot{Q}_j - K(Q,P,t) + \frac{dF}{dt}\left(old, new, t\right)$$

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Type 1:

$$F = F_1(q,Q,t)$$

$$p_j = \frac{\partial F_1}{\partial q_j}\left(q,Q,t\right) \quad P_j = -\frac{\partial F_1}{\partial Q_j}\left(q,Q,t\right) \qquad K = H + \frac{\partial F_1}{\partial t}$$

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Type 2:

$$F = F_2(q,P,t) - Q_i P_i$$

$$p_j = \frac{\partial F_2}{\partial q_j}\left(q,P,t\right) \quad Q_j = \frac{\partial F_2}{\partial P_j}\left(q,P,t\right) \qquad K = H + \frac{\partial F_2}{\partial t}$$

Type 3:

$$F = F_3(p,Q,t) + q_i p_i$$

$$q_j = -\frac{\partial F_3}{\partial p_j}\left(p,Q,t\right) \quad P_j = -\frac{\partial F_3}{\partial Q_j}\left(p,Q,t\right) \qquad K = H + \frac{\partial F_3}{\partial t}$$

Type 4:

$$F = F_4(p,P,t) + q_i p_i - Q_i P_i$$

$$q_j = -\frac{\partial F_4}{\partial p_j}\left(p,P,t\right) \quad Q_j = \frac{\partial F_4}{\partial P_j}\left(p,P,t\right) \qquad K = H + \frac{\partial F_4}{\partial t}$$